International Journal of Theoretical Physics, Vol. 30, No. 10, 1991

Correspondence Principle for the Quantum Net

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Received April 12, 1991

Finkelstein's suggestion for a flexible logic is taken up in the context of his causal net theory. We interpret on the net certain concepts that are first expressed in terms of the canonical "flexible logic" of the macroscopic world, namely, the logic of sheaves over the manifold model, here taken to be flat. From this we infer a correspondence principle in the form of a simple (model-dependent) semantics which translates certain concepts between the purely quantum world of the net and the familiar classical–quantum hybridized world of the macroscopic model. As an application, we derive and solve the reticular version of the massless Dirac equation by analyzing the Dirac operator on the net, where its behavior is easily apprehended.

1. INTRODUCTION

A continuing series of works laid the foundations for a theory which is antecedent to both relativity and quantum theory and thereby achieves their fusion at a deep level (see in particular Finkelstein, 1969*a*,*b*, 1972, 1974, 1987, 1988, 1989, and Finkelstein *et al.*, 1974.) A fundamental step in the construction of this theory is the extension and generalization of the von Neumann interpretation of quantum mechanics in terms of a logic of (quantum) propositions, which is then employed to produce a quantal description of those entities which may be presumed to appear macroscopically as space-time points. The power of this idea is immediately apparent when it is combined with a causality requirement: for then the major kinematical features of relativity emerge spontaneously—this is achieved already in the first paper (Finkelstein, 1969*a*). In subsequent work (Finkelstein, 1987, 1990; Finkelstein and Hallidy, 1990), shortcomings of the original von Neumann-like quantum logics were isolated and repaired, a process culminating in the unveiling of a beautifully symmetric object, the extensor algebra (Barnabei

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0020-7748/91/1000-1273$06.50/0

et al., 1985), which, when interpreted as describing a theory of "quantum" sets, shares many of the properties of ordinary set theory. The substratum of the world is then postulated to be a plexus of primitive spinor like "events" which can combine only according to the laws of quantum extensor logic together with a causal structure, which is also necessarily expressed in terms of the quantum logic. The challenge is to show how ordinary spacetime, with some version of quantum mechanics attached, emerges from this plexus as some kind of limit. A mechanism to account for the initial phase is convincingly argued in Finkelstein (1988, 1989). In this scenario, very roughly speaking, Cooper-like spinor–conjugate spinor pairs form as cooling takes place in the chaotic (but causal) primordial event-vacuum, and a transition is made to a new vacuum, the net. These bosonlike pairs transform as Lorentz vectors and, when aggregated in large numbers, form into vectors in a certain Minkowski space, which apparently corresponds to a local structure on the classical limit manifold, such as the tangent space, or null-cone, at a point. The implementation of this scheme, which involves an intermediate Fock-like algebra, gives a convincing account also of the origin of such basic quantum mechanical items as the Heisenberg communication relations.

The very success of these procedures in giving a good account of the local structure of classical spacetime (plus quantum mechanics) serves to underline the problem of finding a general principle of correspondence mediating between the rigid quantal world of the net and the more familiar (albeit chimerical) world of curvaceous classical spacetime (with quantum fields attached). One source of difficulty may be ascribed to the rigidity of the underlying q-set theory, or logic, itself. Finkelstein has argued the case for a "warped" or flexible logic at this deep level (Finkelstein, 1969*b*, Section III): "If a flexible logic is possible at all, it may be rich enough to account for much more of the phenomena we see at the higher levels than we usually regard as logical in origin."

This remark foreshadows the most important development in modern pure ("classical") logic, which occurred in the year of its publication, namely, the invention by Lawvere and Tierney of topos theory. The motivation for this development lay within pure logic itself, where the notions of classical set theory had been found too rigid. For example, some models of ZF were found by P. J. Cohen in 1963 to falsify certain propositions (the axiom of choice, the continuum hypothesis), while other models had been shown by Gödel in the 1930s to verify these propositions. Topos theory reformulates and generalizes set theory in a categorical setting, the objects in the categories of interest being akin to variable or parametrized sets. Such a category, called a topos, comes equipped with an internal logic which turns out to be strongly typed and intuitionistic. There are geometric aspects to the theory which are surprising from the point of view of pure logic, being absent or invisible in the case of the topos **Set**, the category of ordinary sets, which one may think of as the category of "constant" sets, each object being "parametrized" by a single point. (The existence of these phenomena, and indeed of the whole theory of topoi, would seem to vindicate Finkelstein's "flow follows fracture" dictum in the field of logic, the fracture in this case having been occasioned by Cohen's discoveries.) In short, there are compelling reasons coming not from physics, but from classical logic itself to regard the topos notion as being now more fundamental than the set notion. [For full accounts of topos theory from various viewpoints, see Barr and Wells (1985), Bell (1988), Goldblatt (1984), Johnstone (1977), and Lambek and Scott (1986).]

In this paper we attempt to follow rather informally some of the consequences of implementing Finkelstein's flexible logic suggestion in the context of his causal net theory. If one knew which topos to choose in place of **Set**, this could be effected by replacing classical set-theoretic notions wherever they appear in the construction of the sets ↔ quantum sets correspondence by the appropriate topos-theoretic ones. One would then arrive at the notion of a quantum set theory (i.e., a type of Grassmann algebra) in the chosen topos. Our conjecture is that the true underlying topos (or topos-like structure) may indeed lie beyond **Set**, and that if this is so, its discovery and use, as described above, would represent a significant step in the program, though beyond our present scope. Nevertheless, having the topos notion in mind, it is immediately apparent that the "macroscopic limit" topos is simply the category of sheaves of sets over the macroscopic limit manifold. Our modest goal in this first approach to the topos structure is to start with a flat, noninteracting model of the limit topos and to work back toward the **Set**-based net while keeping as far as possible within the language of this topos (i.e., sheaf theory). By so doing, we hope to have uncovered some traces of topos-like behavior to be used later in the search for a topos more flexible than **Set** upon which to base a more flexible net theory. We find that a judicious choice of model for the limit manifold enables us to infer a correspondence between certain macroscopic classical–quantum notions expressible in sheaf-theoretic terms and notions expressed in the very simple language of the net. (Henceforth we shall use Finkelstein's appellation "cq" in referring to the classical–quantum hybrid world of the limit manifold.) This correspondence is expressed in the form of a few very simple semantical rules whose existence raises the possibility of taking a familiar cq-notion (for example, an equation), of translating it into the language of the net where it is simpler, of carrying out a semantic analysis there (for example, solving the reticular version of the equation), and then of interpreting the result back in the ordinary cq-world. The extent to which the latter result agrees infinitesimally with the usual one, obtained without leaving the cq-world, is presumably one measure of the degree of consistency between the choice of topoi upon which, respectively, the net and the cq-limit are modeled. In our case, these are the minimal choices respectively available: **Set** for the underlying topos, and the topos of sheaves of sets over a Minkowskian manifold for the cq-limit topos. In Section 3, we carry out this test for the massless Dirac equation and find a reasonably good first-order fit.

Since the two topoi just mentioned are the only ones considered here, no knowledge of general topos theory per se will be required of the reader beyond that these categories are in fact topoi. A more fundamental investigation would presumably entail the deduction of a cq-topos as a limiting case of a more primitive underlying structure yet to be discovered. Such an investigation would require a more forceful use of topos theory.

A brief mathematical glossary appears in the Appendix.

2. A MODEL AND ITS CORRESPONDENCES

Finkelstein's quantum version of set theory is constructed by modeling not only the usual set-algebra constructions, but also the more subtle global symmetries (extensionality, intensionality, etc.) of classical set theory and its logic within a slightly modified von Neumann framework. The result is a type of Grassmann algebra which is itself of necessity constructed/defined within the category **Set** of sets. If we were to supplant **Set** in this modeling process by some other topos, then the analogous algebra would presumably be required to incorporate an image of the type structure, etc., and its construction may have to be carried out internally within the very topos whose logic it purports to be quantizing. Alternatively, it may be sufficient to regard this algebra as some kind of meta-object, describable also in **Set**, as an extended version of the extensor algebra. In either case, the corresponding net construction, though capable perhaps of producing a richer cq-structure in the appropriate limit, would undoubtedly look rather different from the **Set**-based one.

In the cq-world, the parameter space over which everything is considered to vary is the spacetime manifold itself, so the choice for the "cq-topos" of the category of sheaves of sets over the manifold is virtually mandated. Within the (intuitionistic) logic of this topos, one can with care construct internal categories of algebraic objects such as groups, rings, modules, etc. With enough care, these turn out to be identical with, respectively, the categories of sheaves of rings, modules, etc. With even more care, the construction of the usual topological fields such as the reals, the complexes, etc., when carried out within the logic, yields, respectively, the sheaves of germs of continuous functions into the usual reals, complexes, etc. [See the works referred to above and their references, particularly Mulvey (1974) and the collection edited by Fourman *et al.* (1979).] Thus, a legitimate first step along the path to the underlying topos would be to express quantized (or, rather, second quantized) cq-notions in sheaf-theoretic language. [The idea of considering a model of classical spacetime, with its geometry, as a topos in which to model quantum phenomena is not new: see Josza's article in Fourman *et al.* (1979), where the emphasis is different from ours.]

If we suppose classical spacetime to be modeled upon some manifold M, then our physics will be assailed by diseases in the form of singularities long before the need to quantize gravity becomes an issue. These singularities in the physics are of course attributable to the pointlike nature of the elements of the substratum. There are (at least) two troublesome issues: first, each point is a potential singularity for some physically relevant quantity, and second, the notion of a point is itself naively classical and should be replaced by some quantized notion. A program to alleviate these problems suggests itself as follows:

1. Consider each point as the locus of some singularity.
2. Try to resolve the singularity in some geometrical sense and globalize the resulting local object.
3. "Quantize."

The resulting object should be interpretable as a "quantum" replacement for spacetime with all the singularities potentially arising from points having been disposed of. There is a standard method of coping with singular points in geometry, namely, "blowing up": roughly speaking, one smears over the offending point a space of directions through it. An obvious choice for the set of directions in our case is the set of rays in the tangent space $T(x)$ at x; or since we are interested in x as a quantized object (and rays in a complex vector space represent maximal information about the object being quantized, x in this case), a better choice is the set of rays in the complexified tangent space at x, the latter being denoted by $T_{\mathbb{C}}(x)$. Then the space of directions is precisely the projective space $\mathbb{P}(T_{\mathbb{C}}(x))$, which is to be "smeared" over the point x. Since we are considering each point of M, the globalized object is nothing but the projectivized complex tangent bundle over M, denoted

$$\pi: \quad \mathbb{P}(T_{\mathbb{C}}) \to M$$

This completes steps 1 and 2 and goes a little way toward accomplishing step 3, since some local quantization has already taken place. To continue further in the direction of step 3, suppose we now second-quantize some complex scalar field defined on $\mathbb{P}(T_{\mathbb{C}})$. Then, according to the view expounded in Selesnick (1983) (see also Mallios, 1990), the states of such a field are represented by sections of a certain complex line bundle on $\mathbb{P}(T_{\mathbb{C}})$. Now it is standard that the line bundle on $\mathbb{P}(T_{\mathbb{C}})$, denoted $\mathcal{O}(1)$ by geometers (the twisting sheaf of Serre, the dual of the universal bundle) generates the group of complex line bundles on this space. (We do not distinguish between vector bundles and their sheaves of germs of sections.) Thus, any set of quantized complex fields can be expressed in terms of functions of sections of tensor powers of $\mathcal{O}(1)$: in this sense $\mathcal{O}(1)$ and its tensor powers $\otimes^n\mathcal{O}(1)$ [conventionally written $\mathcal{O}(n)$] play the role of quantized coordinates, the entities which coordinatize or parametrize the fields. In this view, the sections of $\mathcal{O}(1)$ itself represent single quanta of the basic bosonlike unit of "coordinatization," and the sections of $\mathcal{O}(n)$ accordingly represent ensembles of these quanta. Now there is a sheaf-theoretic construction which allows us to view these sheaves from the perspective of the classical spacetime M, namely, the pushout along π. Thus, $\pi_*\mathcal{O}(n)$, the pushout of $\mathcal{O}(n)$, is a sheaf over M whose sections in a sense represent ensembles of n "coordinate" quanta tied to the spacetime itself. However, it is a standard result that

$$\pi_*\mathcal{O}(n) \cong \circledS^n T_{\mathbb{C}}$$

where $\circledS$ denotes symmetric product. [See Hartshorne (1977), Proposition 7.11, for a proof in the algebraic category. The symmetric nature of this object arises from the fact, pointed out in Selesnick (1983), that any tensor power of a line bundle is necessarily symmetric.] The Fock sheaf

$$\bigoplus_{n\geq 0} \pi_*\mathcal{O}(n) \cong \bigoplus_{n\geq 0} \circledS^n T_{\mathbb{C}}$$

[$\mathcal{O}(0)$ denoting the trivial line bundle] is thus a globalization of the $\mathbb{C}$-polynomial algebra over the generators of the complexified tangent bundle; that is, the fiber over a point x in M is the complex polynomial algebra over a basis of the complexified tangent space at that point and is isomorphic with the appropriate net algebra of Finkelstein (1989). The latter represents the quantized version of the future null-cone at x in Finkelstein's theory; thus we have established contact with that theory by working backward from the cq-model. This provides a basis upon which, using rather heuristic arguments, we shall erect a crude semantic correspondence between (**Set**-based) q-notions and cq-notions. (Whether or not we have now accomplished step 3 is debatable.)

We choose and fix a net element ξ containing an even number of spinor–conjugate spinor pairs and consider the basic event pair succeeding ξ, namely $\delta^{\Sigma\sim} \vee \langle \delta^{\Sigma} \vee \xi |$, where Σ denotes a basic spinor, either $\uparrow$ or $\downarrow$, $\delta^{\Sigma} = \langle \Sigma |$, and the other notation is as in Finkelstein (1989), except that we denote conjugate spinors with a tilde, reserving asterisks for linear duals and sheaf pullbacks. These second-grade plexors in Hermitianized and symmetrized form, when bracketed by $\langle \cdot \mid$, correspond to the generators of the net algebra based at ξ, which in turn correspond as above to local sections of the complexified tangent bundle of our model M at the point in M corresponding to ξ (i.e., the equivalence class of "paths" leading to ξ in the net). From the viewpoint of a putative variable q-logic, we may interpret this correspondence roughly as follows. Being itself a first-grade plexor, ξ is the "name" of a path in the net. The naming of this path (via the application of the $\langle \cdot \mid$ operator) fixes it within its q-"type," and hence it can now represent a point in M. Points in a space have a sheaf-theoretic interpretation for which some terminology is required. Let us denote by **Shv**(X) the category of sheaves of sets over the topological space X. If Y is a topological space and $f: X \to Y$ a continuous function, then there exists a pair of adjoint functors

$$(f_*, f^*): \quad \mathbf{Shv}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbf{Shv}(Y)$$

given respectively by pushout and pullback along f. In topos theory parlance, such a pair constitutes a "geometric morphism." Now denote by P the one-point space, and let x be any point in M. Then the inclusion $\{x\} \hookrightarrow M$ can be regarded as a map $i_x : P \to M$. Then, noting that $\mathbf{Shv}(P) = \mathbf{Set}$, we have a geometric morphism

$$((i_x)_*, (i_x)^*): \quad \mathbf{Set} \to \mathbf{Shv}(M)$$

for each $x \in M$. (For sufficiently nice M's, including Hausdorff ones, these geometric morphisms actually characterize the points of M.) So ξ corresponds to the geometric morphism $((i_\xi)_*, (i_\xi)^*)$ in the cq-logic, where we confuse the net element ξ with the point in M which corresponds to it, when the context is clear. Having fixed the point corresponding to ξ in M within its cq-type (it is of type "geometric morphism"), we now have the succeeding conjoined pairs of spinors (namely, $\delta^{\Sigma^\sim} \vee \langle \delta^\Sigma \vee \xi \mid)$ corresponding to elements of quite a different cq-type, namely, elements of the *set* $(i_\xi)^* T_\mathbb{C}$. This type-like distinction mirrors the one we have imposed at the q-level; for instance, if $\langle \delta^{\Sigma^\sim} \vee \langle \delta^\Sigma \vee \xi \|$ is thought of as a name of a vector (and a net element), then $\delta^{\Sigma^\sim} \vee \langle \delta^\Sigma \vee \xi \mid$ is a vector—not a name of one (and not a net element).

In attempting to specify a local "semantics of correspondence," it will prove helpful to abuse one of the notations used in formal logic: we shall write $[\![expression]\!]$ to denote a cq-interpretation of *expression*. The implicit reading will be something in the nature of $[\![expression]\!]$ = "*expression* when viewed from the cq perspective, i.e., when viewed from afar." We shall regard $[\![\cdot]\!]$ as linear whenever it is sensible to do so, because the linear operations are unified in this theory: for example, the microscopic q-superposition $+$

gives rise to and is identical with, the + of macroscopic cq-vector algebra. Returning to the generating second-grade plexor written above, we note that when viewed from afar, it appears as a vector in the $\Sigma\Sigma^{\sim}$ direction, leading onward from the point corresponding to ξ, lying in the tangent space at this point. This incremental tangent is just derivation in the $\Sigma\Sigma^{\sim}$ direction at the point, and so corresponds to the reticular version of this operator, which we shall denote by $\Delta^{\Sigma\Sigma^{\sim}}$ and discuss in the next section. Let us also write $\Gamma_{\Sigma\Sigma^{\sim}}(\xi)=\delta^{\Sigma^{\sim}}\vee\langle\delta^{\Sigma}\vee\xi|$. The indices are lowered here for the following reason. We have associated $\delta^{\Sigma^{\sim}}\vee\langle\delta^{\Sigma}\vee\xi|$ with the tangent $\partial^{\mu}=\partial/\partial x_{\mu}$ at ξ, where $\mu=\Sigma\Sigma^{\sim}$. The *spacetime* vector corresponding locally to this tangent is then the covariant infinitesimal vector x_{μ} (strictly dx_{μ}), and it is an interpretation of $\Gamma_{\Sigma\Sigma^{\sim}}$ as an operator version of *this* vector which we seek. Considered affinely, once an origin is chosen in the tangent space, the vector ∂^{μ} and its dual dx_{μ} are, for all local physical intents and purposes, identified there. [This identification is here a matter of geometric intuition. There are, however, compelling reasons to formally identify a *general* finite-dimensional vector space with its linear dual in the context of extensor algebra. Cf. Barnabei *et al.* (1985).] Thus, the simultaneous association "from afar" of $\delta^{\Sigma^{\sim}}\vee\langle\delta^{\Sigma}\vee\xi|$ with, on the one hand, a tangent at the point corresponding to ξ and, on the other hand, with the spacetime vector to which this tangent corresponds, can be expressed in the affine form

$$0_{\xi}+[\![\Delta^{\Sigma\Sigma^{\sim}}(\xi)]\!]=[\![\Gamma_{\Sigma\Sigma^{\sim}}(\xi)]\!]$$

where we have identified the point in M corresponding to ξ with the origin 0_{ξ} in its tangent space $(i_{\xi})^{*}T_{\mathbb{C}}$. That is to say, when $[\![\cdot]\!]$ is evaluated in this tangent space, and the point corresponding to ξ is identified with the origin, we have

$$[\![(\Delta^{\Sigma\Sigma^{\sim}}-\Gamma_{\Sigma\Sigma^{\sim}})(\xi)]\!]=0$$

Now we can lift the domain of the interpretation, i.e., the range of $[\![\cdot]\!]$, by noting that the assignment $\xi\rightarrowtail 0_{\xi}$ can be viewed as a (bosonic) state representing vacuous positional information, the coordinate vacuum, and that the operators defined by

$$[\![(\Delta^{\Sigma\Sigma^{\sim}}-\Gamma_{\Sigma\Sigma^{\sim}})]\!][\![\xi]\!]=[\![(\Delta^{\Sigma\Sigma^{\sim}}-\Gamma_{\Sigma\Sigma^{\sim}})(\xi)]\!]$$

apparently annihilate this vacuum. Now it is easy to find a domain in which to interpret this. Consider the Schrödinger representation on $L_2(\mathbb{R}^4)$, for instance: the destruction operators

$$a_{\mu}\equiv\frac{1}{i}\partial^{\mu}-ix_{\mu}$$

annihilate an appropriate vacuum, ω. Following Finkelstein (1988, 1989), let us introduce the reticular time constant $\sqcap$ by expressing the macroscopic time measurement of x^μ in terms of the corresponding discrete dimensionless net measurement (a count of steps, which we denote by $x^{\Sigma\Sigma^\sim}$) as $x^\mu = \sqcap x^{\Sigma\Sigma^\sim}$. Then, identifying a_μ with the operator defined above, we are led to posit the following correspondence (which are "local to ξ"):

C1. $[\![\xi]\!] = \omega_\xi$
C2. $[\![\Delta^{\Sigma\Sigma\zeta}]\!] = -i\sqcap\partial^\mu$
C3. $[\![\Gamma_{\Sigma\Sigma\zeta}]\!] = (i/\sqcap)x_\mu$

The understanding here is that the right-hand sides of the above equations refer to a particular representation of the canonical commutation relations. Owing to the celebrated uniqueness properties of such representations, it is possible to vary the domain of interpretation, which is the underlying vector space itself, without affecting the operator algebraic structure. Thus, one could carry out the interpretation in various Fock space models with ω_ξ taken as the appropriate vacuum and ∂^μ, x_μ defined in terms of the associated creation and destruction operators. In all cases, the algebraic structure will be essentially the same, since the vacuum is a cyclic vector for the algebra generated by these operators.

In the next section, after discussing the operator $\Delta^{\Sigma\Sigma^\sim}$, we present an argument that does not depend upon preordained Schrödinger destruction operators to support C1–C3.

To extend this C1–C3 correspondence, we specialize the model space-time M. We note first that the choice of a complex manifold structure for M would not be inappropriate, for then the tangent bundle (T) would itself be a bundle of complex vector spaces in an intrinsic way. If this choice is agreed upon, then there is a canonical complex model for Minkowski space-time, namely, its conformal compactification realized as the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ of two-dimensional subspaces of $\mathbb{C}^4$. This choice has an important property which, in a sense, globalizes the relation spinor $\times$ conjugate spinor = vector, which lies at the heart of the quantum condensation phenomenon and is presumed to generate the macroscopic manifold structure from the net. Namely,

$$T \cong \tilde{S}^* \otimes S^* \tag{1}$$

where S (respectively $\tilde{S}$) denotes the appropriate spinor (respectively conjugate spinor) bundle on $M \equiv \mathrm{Gr}_2(\mathbb{C}^4)$ (Manin, 1988). Thus, the previous association of $\delta^{\Sigma^\sim} \vee \langle \delta^\Sigma \vee \xi|$ with an element in $(i_\xi)^*T$ (i.e., a tangent at ξ) now becomes an association [via the isomorphism of equation (1)] with an element of $(i_\xi)^*(\tilde{S}^* \otimes S^*)$. This association can then be dissected further, because sheaf pullback commutes with tensor product

$$(i_\xi)^*(\tilde{S}^*\otimes S^*)\cong(i_\xi)^*(\tilde{S}^*)\otimes(i_\xi)^*(S^*)$$

Thus, the vector–spinor conjunction $\langle\delta^\Sigma\vee\xi|$ (which is the name of a spinor) corresponds to an element in $(i_\xi)^*(S^*)$; that is, a spinor at the point corresponding to ξ. In more colloquial, nontoposophical parlance, we may interpret $\langle\delta^\Sigma\vee\xi|$ as an analogue of the pairing (δ^Σ, ξ) which specifies an element in the fiber of the bundle S^* over the point corresponding to ξ. (With δ^Σ replaced by other inputs, including the $\delta^{\Sigma^\sim}$, this would appear to represent a general interpretation of at least some of the paraphernalia associated with the notion of a gauge interaction and, as such, agrees with that of Finkelstein.) Sufficiently local successors to $\langle\delta^\Sigma\vee\xi|$, namely $\delta^{\Sigma^\sim}\vee\langle\delta^\Sigma\vee\xi|$, then correspond in this sense to fiber elements of the form $(\delta^{\Sigma^\sim}\otimes\delta^\Sigma, \xi)$ in the bundle $\tilde{S}^*\otimes S^*$. The foregoing interpretation lies in, or is derived from, the cq-topos domain and, as such, must now be related to the domain used for $[\![\cdot]\!]$ or an extension of the latter. Before attempting this, it is worth noting that again there is a splitting up of types as we move from the plexor domain to the cq-topos domain. Thus, $\vee$ sometimes corresponds to the tensor product of sheaves and at other times does not, depending upon its context within a plexor.

Now the domain for $[\![\cdot]\!]$ has been taken as some vector space (accommodating a representation of the CCR). To extend it consistently to plexors of "fiber type," namely, $\langle s\vee\xi|$, where s stands for δ^Σ or $\delta^{\Sigma^\sim}$, we seek a vector space which accommodates an analogue of pairs $([\![s]\!], [\![\xi]\!])$. Recalling that $[\![\cdot]\!]$ is linear, we are led to adopt the following slight generalization:

C4. $[\![\langle s\vee v|]\!]=[\![s]\!]\otimes[\![v]\!]$

where s denotes a plexor of spinor type, v a plexor of vector type, and where the domain accommodating the right-hand side is, of course, implicitly determined.

This completes our rather minimal local correspondence semantics for this model. We have in fact invoked two different kinds of semantics: an informal one, whose domain of interpretation is the language of the cq-topos and/or its relatives, and a more formal one derived from it, namely, C1–C4.

The rest of the paper is devoted to a study of the massless Dirac equation using a combination of these semantics.

3. THE DIRAC EQUATION

We shall interpret the ordinary massless cq-Dirac equation on the net by pulling back (via the informal semantics) from a sheaf-theoretic version of the Dirac operator available in our cq-model. A "semantic" or, to be less pretentious, verbal interpretation is then available, thanks to the limited language employed by the net. Using some elementary q-logic, this is used to concoct a simple solution to the reticular version of the Dirac equation, which is then interpreted back in the cq-world with the aid of the formal semantics.

Reverting to equation (1), we note that

$$\Omega^1 \cong T^* \cong S \otimes \tilde{S} \tag{2}$$

where Ω^1 denotes the bundle of 1-forms on $M = \mathrm{Gr}_2(\mathbb{C}^4)$. Consequently, we have two linear sheaf maps

$$\sigma: \quad \Omega^1 \otimes S^* \to \tilde{S}$$

$$\tilde{\sigma}: \quad \Omega^1 \otimes \tilde{\mathrm{S}}^* \to \mathrm{S}$$

given by the appropriate contraction when the isomorphism of equation (2) is invoked. Consequently, each generator of Ω^1 induces a linear map $\tilde{S}^* \oplus S^* \to \tilde{S} \oplus S$. With an appropriate choice of basis, these maps correspond to the analogues of the Dirac matrices (Manin, 1988). To transfer these notions back to the net, we note that the generators of Ω^1 correspond by duality to the "tangents" $\delta^{\Sigma^\sim} \vee \langle \delta^\Sigma \vee \xi|$ at the point corresponding to ξ. The corresponding Dirac maps, denoted by $\gamma_{\Sigma\Sigma^\sim}$, must then, according to our informal semantics [in which the leftmost $\vee$ in the plexor above corresponds to the $\otimes$ in equation (2)], act fiberwise on plexors of "fiber-type" $\langle \delta^{\Sigma^\intercal} \vee \xi|$ as follows:

$$\gamma_{\Sigma\Sigma^\sim}(\langle \delta^{\Sigma^\intercal} \vee \xi|) = \langle \delta^{\Sigma^\sim} | \delta^{\Sigma^\intercal} \rangle \langle \delta^\Sigma \vee \xi| \tag{3}$$

or, more generally, with ξ replaced by a general plexor of vector type. In this definition, we have identified the "fiber" of S with its dual in an obvious way, and we have used the standard inner product notation.

In searching for a plexorial version, $\Delta^{\Sigma\Sigma^\sim}$, of the cq operator ∂^μ, an operator we have already named in C2 but not yet defined, we must confront a fundamental difference between the q and cq worlds. This resides in the fact that "observables," etc., of the q world should be associated with non-commuting objects, whereas the c in cq inevitably injects some commutativity into underlying cq notions, even though these are partially hidden by the imposition of q structure (i.e., quantization, second quantization). [See Mallios (1990) and Selesnick (1983) for an attempt to uncover and make explicit this usually suppressed commutative structure in the context of second quantization.] Thus, in the case at hand, ∂^μ acts on functions defined on neighborhoods of spacetime points (elements of a commutative algebra) and leaves the points themselves intact. The only observables available to the pure net-theorist, however, are the net plexors themselves, which combine according to the noncommutative rules of q-logic. Consequently, we must find a local analogue (at, or depending in some local manner upon, ξ) of an algebra of differentiable functions built out of these elements. The considerations of Section 1 lead us to choose, with Finkelstein, the algebra $\mathbb{C}[\Gamma_\sigma]$ of complex polynomials in the indeterminates Γ_σ ($\sigma=\Sigma\Sigma^\sim$). Then $\Delta^{\Sigma\Sigma^\sim}$ must be ordinary partial differentiation with respect to $\Gamma_{\Sigma\Sigma^\sim}$, an operation which can now be defined on any plexor of the form $p(\Gamma_\sigma)(\xi)$ by using the product rule, where p is a polynomial in the Γ_σ and the symbol $\langle$, whose order and syntax must be respected. The trailing strings of $|$'s, which are determined by the simple syntax of brackets, are simply ignored in this notation. Thus,

$$\Delta^{\Sigma\Sigma^\sim}(p(\Gamma_\sigma)(\xi)) \equiv (\Delta^{\Sigma\Sigma^\sim} p(\Gamma_\sigma))(\xi)$$

[Note, in particular, that $\Delta^{\Sigma\Sigma^\sim}\xi=\Delta^{\Sigma\Sigma^\sim}\Gamma^0_{\Sigma\Sigma^\sim}(\xi)=0$. We emphasize that this definition of $\Delta^{\Sigma\Sigma^\sim}$ depends upon ξ; that is, it is local to ξ.]

This operator can now be applied to net elements (of vector type) succeeding ξ. The result, in general, is a superposition of net elements: the net element suffers the destruction of certain pairs of nodes, and the resulting plexors are superposed. Thus, $\Delta^{\Sigma\Sigma^\sim}$ acts on the net in an anticausal direction counter to the direction in which nodes are created by the Γ_σ. This is precisely contrary to the analogous classical behavior of the operator ∂^μ, which, when considered as a tangent vector on the manifold, points in the *same* direction as the direction of increase of x_μ—a rather tautological statement. This now accounts for the i in the expressions on the right-hand sides of C2 and C3. For, suppose that instead of appealing to the Schrödinger representation in Section 1, we were to independently adopt the assignment $[\![\Gamma_{\Sigma\Sigma^\sim}]\!]=\alpha x_{\Sigma\Sigma^\sim}$ for some α. Then, since the incrementation is assumed to be taking place in the tangent space, $x_{\Sigma\Sigma^\sim}$ can be identified (as in the discussion following the definition of $\Gamma_{\Sigma\Sigma^\sim}$ in Section 2) with the corresponding derivation $\partial^{\Sigma\Sigma^\sim}$ regarded now as a vector in the tangent space, not as an operator. The reticular increment induced by $\Gamma_{\Sigma\Sigma^\sim}$ then corresponds with the tangent $\alpha\partial^{\Sigma\Sigma^\sim}$. On the other hand, $[\![\Delta^{\Sigma\Sigma^\sim}]\!]$ must correspond in the limit to the act of differentiation with respect to the spacetime *variable* corresponding to $[\![\Gamma_{\Sigma\Sigma^\sim}]\!]$; that is, it corresponds to $(1/\alpha)\partial^{\Sigma\Sigma^\sim}$ while maintaining the property, when considered as a tangent, of simultaneously pointing in the direction *opposite* to that of the tangent corresponding to the reticular increment itself, namely, $\alpha\partial^{\Sigma\Sigma^\sim}$. Thus, $1/\alpha=-\alpha$, and $\alpha=\pm i$. Choosing the plus sign now corresponds to specifying that the direction of incrementation is as expected. Readers who find these arguments convincing may now reconstruct the Schrödinger vacuum, C1, etc., *ab initio*.

An analogue $\not{\Delta}_N$ of the Dirac operator can be defined on the net elements based at ξ as follows:

$$\not{\Delta}_N(\langle\delta^{\Sigma\tilde{1}} \vee \langle p(\Gamma_\sigma)(\xi)\|) \equiv \gamma_{\Sigma\Sigma^\sim}(\langle\delta^{\Sigma\tilde{1}} \vee \Delta^{\Sigma\Sigma^\sim} p(\Gamma_\sigma)(\xi)|)$$

summing over repeated index-pairs as usual. In general, the image is not a net element, but near ξ, the behavior of $\not{\Delta}_N$ is particularly pleasant. Namely, with $\sigma = \Sigma_1\Sigma_{\tilde{2}}$ we have

$$\begin{aligned} \not{\Delta}_N(\langle\delta^{\Sigma\tilde{0}} \vee \langle\Gamma_\sigma(\xi)\|) &= \gamma_\sigma(\langle\delta^{\Sigma\tilde{0}} \vee \xi|) \\ &= \langle\delta^{\Sigma\tilde{2}}|\delta^{\Sigma\tilde{0}}\rangle\langle\delta^{\Sigma_1} \vee \xi| \end{aligned} \tag{4}$$

so we arrive back in the net, or die. This local effect of $\not{\Delta}_N$ lends itself to the following interpretation: Conjugate spinor successors to the net elements $\langle\Gamma_\sigma(\xi)|$ [which, from afar, look like elements of the fiber of $\tilde{S}^*$ over the point corresponding to $\langle\Gamma_\sigma(\xi)|$] provide instructions on whether to move back to the predecessor of $\langle\Gamma_\sigma(\xi)|$ or die (give 0) according to the rule given above. (Note that this is an anticausal operation.)

So, to construct a plexorial solution Φ_N to the massless reticular Dirac equation $\not{\Delta}_N\Phi_N = 0$, we argue as follows. Suppose a plexor could be found to represent the universal *absence* of one or the other inputs of type $\Sigma^\sim$, a sort of defect. Then insertion of this object into the net as an input (creating a successor) to certain nodes near ξ should yield the null instruction when the resulting plexors are properly superposed according to the dictates of causality. Then, when $\not{\Delta}_N$ is applied to this object, and it is propagated back, it vanishes. The "proper superposition," which is where quantum and causal structure meet, is the key step here.

A simple program to create such a defect and insert it after the net element η, which combines elements of both classical and q-logic, might run as follows:

begin
    destroy one of the inputs $\downarrow^\sim$ or $\uparrow^\sim$: that is,
        destroy an element of the set $\{\downarrow^\sim\} \cup \{\uparrow^\sim\}$;
    form a plexor which somehow describes this act of destruction;
    insert this plexor after η
end

Now we attempt to fully quantize this program using Finkelstein's rules and remembering to superpose appropriate alternatives, since quantum interference must always be occurring in the pure q-world. (Feynman and Hibbs, 1965, Chapter 1.)

```
begin
    find the appropriate destruction operators for ⟨↓~| ∨ ⟨↑~|;
    find plexor by applying resulting operators to ⟨↓~| ∨ ⟨↑~|
        and superposing the alternatives;
    make resulting plexor the causal successor to η,
        superposing appropriate alternatives
end
```

To execute this program, note first that the appropriate destruction operators in the first step are precisely the ones associated with the fermion Fock space based on $\tilde{S}^*$, which we denote by $a(\downarrow^{\sim})$ and $a(\uparrow^{\sim})$. [Thirring (1983), p. 21, but beware of the misprint on line 21.] Applying these, we obtain

$$a(\downarrow^{\sim})(\langle\downarrow^{\sim}| \vee \langle\uparrow^{\sim}|) = \langle\uparrow^{\sim}| \equiv \delta^{\uparrow^{\sim}}$$

$$a(\uparrow^{\sim})(\langle\downarrow^{\sim}| \vee \langle\uparrow^{\sim}|) = -\langle\downarrow^{\sim}| \equiv -\delta^{\downarrow^{\sim}}$$

So the plexor we seek is their superposition: $\delta^{\uparrow^{\sim}} - \delta^{\downarrow^{\sim}}$.

For the last step, let us first take $\eta = \xi$: this gives the trivial solution $\langle(\delta^{\uparrow^{\sim}} - \delta^{\downarrow^{\sim}}) \vee \xi|$. Next, let $\eta = \langle\Gamma_{\Sigma\Sigma^{\sim}}(\xi)|$. If the defect is to appear after one of these events to be propagated backward by $\blacktriangle_N$, then there are only two proper causal alternatives, namely, that event and its companion on the same causal branch of the binary tree emanating from ξ, since the other branch is causally independent. That is, only one of the pairs $(\uparrow\downarrow^{\sim}, \uparrow\uparrow^{\sim})$, $(\downarrow\downarrow^{\sim}, \downarrow\uparrow^{\sim})$ needs to enter the superposition at this stage; each choice gives a different kind of defect.

Now, since it is apparent that the defect represents some kind of causal or anticausal disturbance in the normal progress of what macroscopically represents certain of the coordinates, it seems that the pair (chosen above) which contains the correspondent of the macroscopic *time* coordinate would be the one which, in the cq-limit, has a chance of producing a null or time-like quantum, whereas the other pair would be associated with something tachyonlike. Let us assume then that the first-named pair contains the time correspondent. Then the last step yields the plexor

$$\Psi_N \equiv \langle(\delta^{\uparrow^{\sim}} - \delta^{\downarrow^{\sim}}) \vee (\langle\Gamma_{\uparrow\uparrow^{\sim}}(\xi) + \Gamma_{\uparrow\downarrow^{\sim}}(\xi)|)|$$

We verify that this Ψ_N does indeed satisfy the reticular Dirac equation [*cf.* equation (4)]

$$\begin{aligned}\blacktriangle_N \Psi_N &= (\gamma_{\uparrow\uparrow^{\sim}} + \gamma_{\uparrow\downarrow^{\sim}})(\langle(\delta^{\uparrow^{\sim}} - \delta^{\downarrow^{\sim}}) \vee \xi|)\\ &= \langle\delta^{\uparrow} \vee \xi| - \langle\delta^{\uparrow} \vee \xi|\\ &= 0\end{aligned}$$

These two local solutions are the only (nontachyonic) ones produced by this direct defect insertion mechanism. They represent the alternatives: a defect appears at ξ; a defect appears at either $\langle\Gamma_{\uparrow\uparrow^\sim}(\xi)|$ or $\langle\Gamma_{\uparrow\downarrow^\sim}(\xi)|$; so to properly describe the defect, we should superpose these two solutions. The solution so obtained may now be interpreted in the cq-world of our model using our semantics if a minor adjustment is made. Namely, since the expression $\langle\Gamma_{\Sigma\Sigma^\sim}(\xi)|$ has no interpretation in our semantics (which is local to ξ, this point being a successor), we simply drop the offending bracket from Ψ_N. The resulting plexor is no longer tied as firmly to the net, but is still a solution to the general massless plexorial Dirac equation. Written in full, this solution is

$$\Phi = \langle(\delta^{\uparrow^\sim} - \delta^{\downarrow^\sim}) \vee (\xi + \Gamma_{\uparrow\uparrow^\sim}(\xi) + \Gamma_{\uparrow\downarrow^\sim}(\xi))|$$

So

$$\begin{aligned}
[\![\Phi]\!] &= [\![\delta^{\uparrow^\sim} - \delta^{\downarrow^\sim}]\!] \otimes [\![\xi + \Gamma_{\uparrow\uparrow^\sim}(\xi) + \Gamma_{\uparrow\downarrow^\sim}(\xi)]\!] && \text{by C4}\\
&= [\![\delta^{\uparrow^\sim} - \delta^{\downarrow^\sim}]\!] \otimes ([\![\xi]\!] + [\![\Gamma_{\uparrow\uparrow^\sim}]\!][\![\xi]\!] + [\![\Gamma_{\uparrow\downarrow^\sim}]\!][\![\xi]\!])\\
&= [\![\delta^{\uparrow^\sim} - \delta^{\downarrow^\sim}]\!] \otimes \left(1 + \frac{i}{\hbar}(x_{\uparrow\uparrow^\sim} + x_{\uparrow\downarrow^\sim})\right)\omega_\xi && \text{by C1–C3}
\end{aligned}$$

We can legitimately write this last expression as

$$\left(1 + \frac{i}{\hbar}(x_{\uparrow\uparrow^\sim} + x_{\uparrow\downarrow^\sim})\right)([\![\delta^{\uparrow^\sim} - \delta^{\downarrow^\sim}]\!] \otimes \omega_\xi)$$

if we specify further that the $\otimes$ in C4 is to be taken over the algebra generated by the commuting operators x_μ, which generate a subspace of the representation accommodating the vector part of the image of $[\![\cdot]\!]$ by acting on ω_ξ, and which have a natural action on the spinor part inherited from the module structure on the associated sheaf. Now, choosing the basis $[\![\delta^{\uparrow^\sim}]\!] \otimes \omega_\xi$, $[\![\delta^{\downarrow^\sim}]\!] \otimes \omega_\xi$ (in this order) in the subspace spanned by them and letting the expectation values $x_{\Sigma\Sigma^\sim} \to 0$, we may write the classical limit locally as

$$[\![\Phi]\!] \approx \begin{pmatrix}1\\0\end{pmatrix} \exp \frac{i}{\hbar}(x_{\uparrow\uparrow^\sim} + x_{\uparrow\downarrow^\sim}) + \begin{pmatrix}0\\-1\end{pmatrix} \exp \frac{i}{\hbar}(x_{\uparrow\uparrow^\sim} + x_{\uparrow\downarrow^\sim}) \tag{5}$$

If we take $\uparrow\uparrow^\sim = 0$ and $\uparrow\downarrow^\sim = 3$, then

$$\begin{aligned}
\frac{1}{\hbar}(x_0 + x_3) &= t - z, \qquad \text{in net units,}\\
&= p_\mu x^\mu
\end{aligned}$$

where $p^\mu = (1, 0, 0, 1)$ and $x^\mu = (t, x, y, z)$, in which case, with $\hbar = c = 1$, the first summand on the right-hand side of equation (5) is a solution to the Weyl equation

$$(i\partial_t - \boldsymbol{\sigma} \cdot \mathbf{p})u = 0$$

and so represents a negative-energy massless Dirac fermion. The other summand is a solution to

$$(i\partial_t + \boldsymbol{\sigma} \cdot \mathbf{p})u = 0$$

and so represents the corresponding particle with opposite spin.

Thus, in the continuum limit, we get a superposition of the two possible states of a massless antifermion moving along one of the axes, at least locally. [The fact that antiparticles emerge from our "semantic" argument should not be surprising in view of the anticausal interpretation we have given to the Δ operator: the positive-energy plexor analogue with signs of $\Gamma_{\Sigma\Sigma^{\sim}}(\xi)$ reversed is, of course, also a solution.]

It is worth remarking that since the plexorial solution is properly to be regarded as lying in a space of qets, it is a *field*, in Finkelstein's sense.

(Nonlocal solutions to the reticular equation can be found by analogous arguments.)

4. CONCLUSIONS

A significant feature of the cq-topos, topoi in general, and, indeed, of the world itself is the profusion of types absent from **Set**. It is possible that this proliferation of types is an artifact of the q→cq transition itself, a breaking of the type-symmetry which may exist in the pristine q-world, in which case **Set** may well be good enough. Indeed, we have found that at least for our interpretation of the massless Dirac equation, reasonable consistency obtains when it is assumed that the theory of the **Set**-based net gives rise in the cq-limit to the topos of sheaves of sets over a model of Minkowski space; and it seems likely that the **Set**-based net will also prove capable of generating gravitational curvature viewed on the net as a flow of one type of spin. But whether **Set**-based models can produce the full panoply of the world and its "types" (gauge interactions, fermion generations, possible Higgs fields, etc.) is another question, and one which will undoubtedly require a large effort to settle.

In the meantime, a search for other possible foundational topoi (or relatives of topoi) would seem to be in order, in which case analogues of the pseudosemantics given here for curved interacting models may prove useful.

APPENDIX: SOME MATHEMATICAL TERMINOLOGY

Presheaves [Bredon (1967), or the works on topos theory already cited]

A *presheaf* A on a topological space X is an assignment to each open $U \subset X$ of a set $A(U)$ and to each pair $U \subset V$ of open sets of a function $\rho_{UV}: A(V) \to A(U)$, called "restriction," in such a way that

P1. $\rho_{U,U} = 1$
P2. $\rho_{U,V}\rho_{V,W} = \rho_{U,W}$, when $U \subset V \subset W$

Thus, a presheaf imitates the behavior of a family of functions defined on open sets. Indeed, starting with a presheaf A on X, one may produce a topological space $\mathscr{A}$, and for each open $U \subset X$, a map

$$\sigma_U: A(U) \to \Gamma(U, \mathscr{A})$$

(where the latter denotes a certain family of continuous functions), which commutes with restrictions in an obvious way. Specifically for $x \in X$, let

$$\mathscr{A}_x \equiv \varinjlim \{A(U) \mid U \ni x\}$$

$$\equiv \dot{\bigcup} \{A(U) \mid U \ni x\} / \tilde{x}$$

where for $f \in A(U)$ and $g \in A(V)$, $f_{\tilde{x}} g$ means that

$$\rho_{W,U \cap V}(f) = \rho_{W,U \cap V}(g)$$

for some open $W \subset U \cap V$ such that $x \in W$. Then, as a set,

$$\mathscr{A} = \dot{\bigcup_x} \mathscr{A}_x$$

and we define a topology on $\mathscr{A}$ as follows: For $f \in A(U)$ and $x \in U$, let $[f]_x$ denote the $\tilde{x}$ equivalence class of f (the "germ" of f at x). Then the topology on $\mathscr{A}$ is generated by basic open sets of the form $\{(x, [f]_x) \mid x \in U\}$. With $\Gamma(U, \mathscr{A})$ defined to be the set of continuous *sections* of this structure [i.e., continuous maps $s: U \to \mathscr{A}$ such that $s(x) \in \mathscr{A}_x$], σ_U is defined by $\sigma_U(f)(x) = [f]_x$, $x \in U$.

Sheaves [Bredon (1967), or the works on topos theory already cited]

In the discussion above, the map $\pi: \mathscr{A} \to X$ given by $\pi(x, [f]_x) = x$ can be shown tc be a local homeomorphism. In general, a local homeomorphism $p: Y \to X$ is called a *sheaf*. The fibers (which are easily seen to be relatively discrete) are called the *stalks* of the sheaf. (Usually there is additional algebraic structure in the stalks with concomitant assumptions concerning the "horizontal" continuity of the algebraic operations, but we shall ignore this possibility here.) Thus, a presheaf A gives rise to (or *generates*) a sheaf $\mathscr{A}$. This process is often referred to as the "sheafification" of A. Returning to the sheaf p, let $\Gamma(U, Y)$ denote as before the set of continuous sections of p over U. Then the assignments $U \mapsto \Gamma(U, Y)$, with ordinary restriction, constitute a presheaf which satisfies in addition the following "collation" properties:

S1. If $U = \bigcup_\alpha U_\alpha$ with U_α open, and $s, t \in \Gamma(U, Y)$ are such that $s \mid U_\alpha = t \mid U_\alpha$ for all α, then $s = t$.
S2. Let $\{U_\alpha\}$ be a collection of open sets in X, and let $U = \bigcup_\alpha U_\alpha$. If $s_\alpha \in \Gamma(U_\alpha, Y)$ are given such that $s_\alpha \mid U_\alpha \cap U_\beta = s_\beta \mid U_\alpha \cap U_\beta$ for all α, β, then there exists an element $s \in \Gamma(U, Y)$ with $s \mid U_\alpha = s_\alpha$ for each α.

Reverting to the presheaf A, let us suppose that it satisfies the analogues of S1 and S2. Then it can be shown that the σ_U are in fact isomorphisms. Thus, a presheaf satisfying S1 and S2 can be realized precisely as the presheaf of continuous sections of a sheaf; consequently, the distinction between such presheaves and "spatial" sheaves is often not made. We note that as a consequence of the isomorphisms, a sheaf may be regenerated from its presheaf of sections.

The sheaf $\mathscr{F}$ of germs of continuous sections of a locally trivial n-dimensional k-vector bundle F, $k = \mathbb{C}, \mathbb{R}, \ldots$, has the property that there exists an open covering $\{U\}$ of X such that for each U,

$$\Gamma(U, \mathscr{F}) \cong \bigoplus^n C(U, k)$$

where $C(\cdot, k)$ denotes continuous functions into k. One observes that this sheaf preserves the action of the bundle's Čech cocycles so that the bundle may be reconstructed from its sheaf of germs of sections and, moreover, from the algebraic structure of the set of sections; these structures are essentially equivalent. [For a very general treatment of geometry from this point of view, see Mallios (1991).]

The family of presheaves over a topological space X, with morphisms defined in an obvious way, constitutes a category we denote by **Pre**(X). Any continuous function $f\colon Y \to X$, with Y another topological space, induces a pair of (adjoint) functors:

$$f_*: \quad \mathbf{Pre}(Y) \to \mathbf{Pre}(X)$$

called *direct image* or *pushout*, defined for a presheaf A on Y by $(f_* A)(U) = A(f^{-1}(U))$ for U open in X, and

$$f^*: \mathbf{Pre}(X) \to \mathbf{Pre}(Y)$$

called *inverse image* or *pullback*, defined for a presheaf B on X and V open in Y by

$$(f^*B)(V) = \varinjlim\{B(U)\,|\,f(V) \subset U \text{ and } U \text{ open in } X\}$$

These functors can be defined similarly on the respective full subcategories of sheaves, the first definition remaining unchanged, but the second requiring sheafification.

Topoi (See references already cited)

A category **E** is an *elementary topos* if it satisfies:

T1. **E** is finitely complete.
T2. **E** has exponentiation.
T3. **E** has a subobject classifier.

The second condition means roughly that for every pair a, b of objects, there is an object b^a corresponding to Hom(a, b) in such a way that canonical isomorphisms Hom$(c \times a, b) \cong$ Hom(c, b^a) obtain for all c.

The third condition says essentially that there exists an object Ω in **E** such that for any object a, the subobjects of a can be canonically realized in Hom(a, Ω) and conversely. That is, sub$(a) \cong$ Hom(a, Ω) in a canonical fashion.

Examples include **Set**, **Shv**(X) for X a topological space and the category $\mathbf{Set}^{\mathbf{C}^{op}}$ of contravariant functors from any small category **C** to **Set**, an example which includes **Pre**(X).

It follows from T1 that any topos has a terminal object, usually denoted by 1; this is the object specified up to isomorphism by the property that for any object a, there is a unique morphism $a \to 1$. Thus, any object is "fibered" over 1, and any morphism respects this fibering. In **Set**, 1 is the one-point space, so this fibering is rather uninteresting. Note, however, that sub$(1) \cong \Omega = \mathbf{2}$, the two-point space (a Boolean algebra). In a general topos **E**, we still have sub$(1) \cong$ Hom$(1, \Omega) \cong \Omega$, but the latter may have a more complex structure being interpretable within **E** as a complete Heyting algebra. Thus, the fibering over 1 may be more interesting. This is one sense in which objects of **E** resemble "parametrized" sets.

An internal logic may be associated with any topos which turns out in general to be a type theory taking truth values in Ω. [For a significant application of this logic in the case of **Shv**(X), see Mulvey (1974).]

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